# Euler Equation and Bernoulli's Theorem

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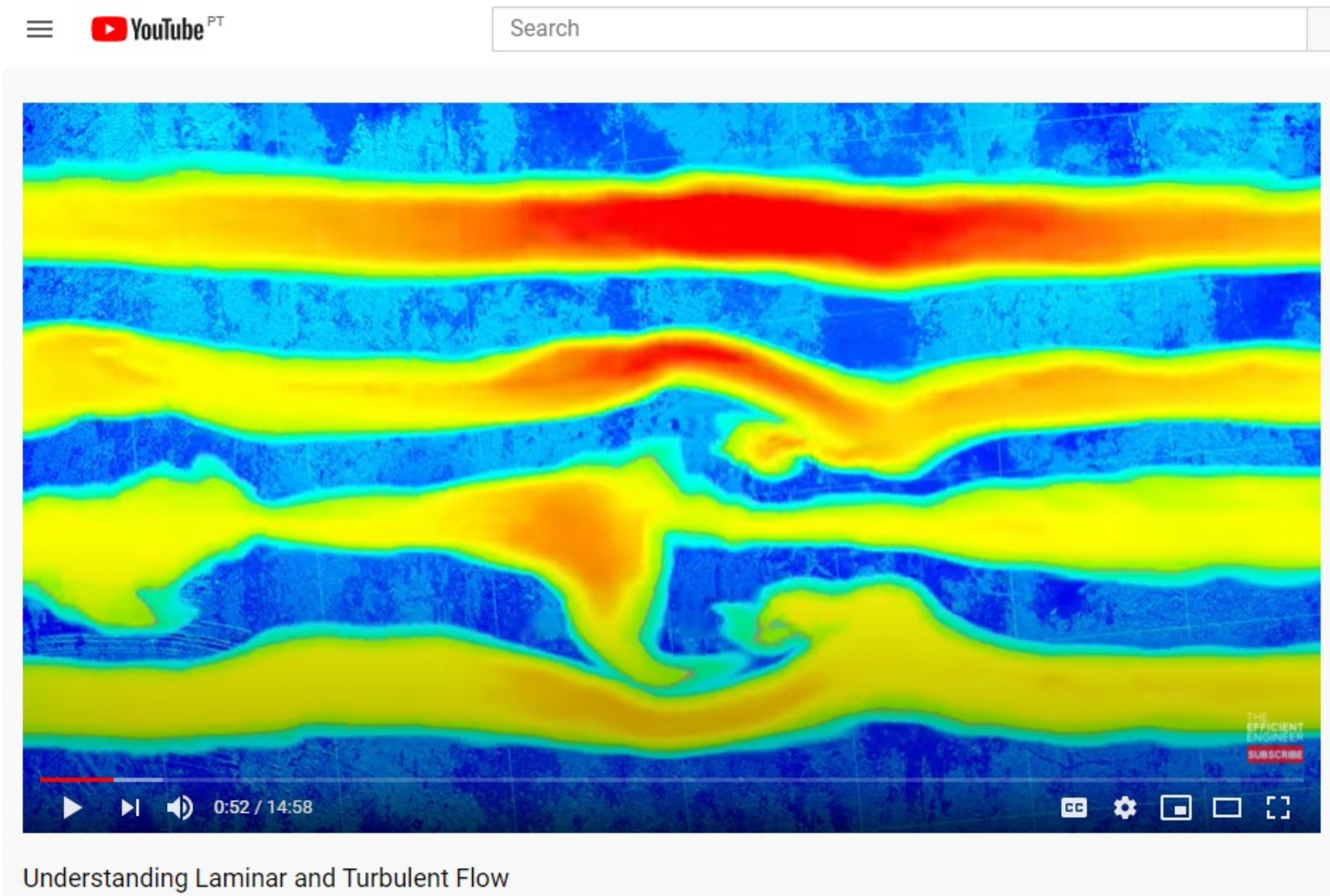
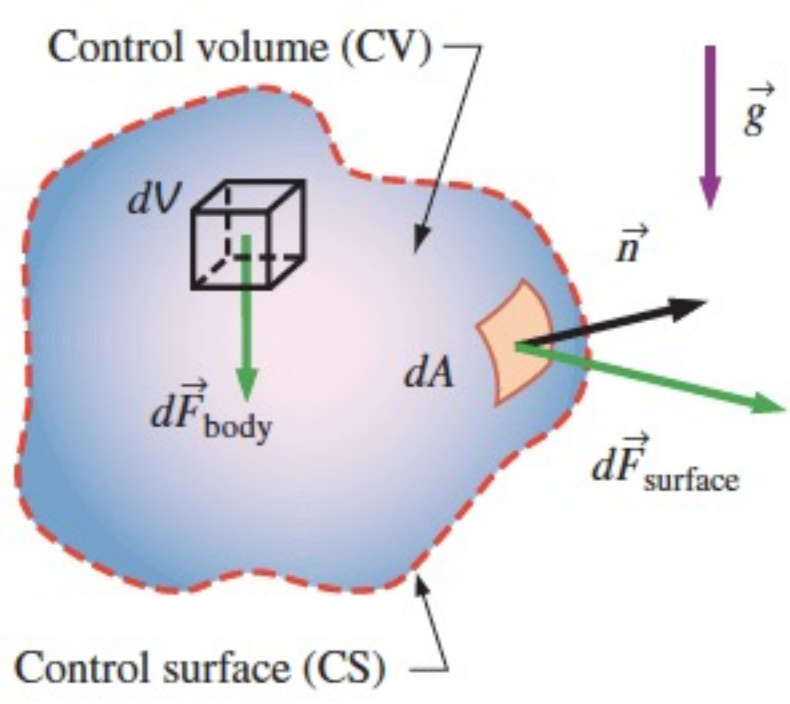
https://www.youtube.com/watch?v=9A-uUG0WR0w



# Overview

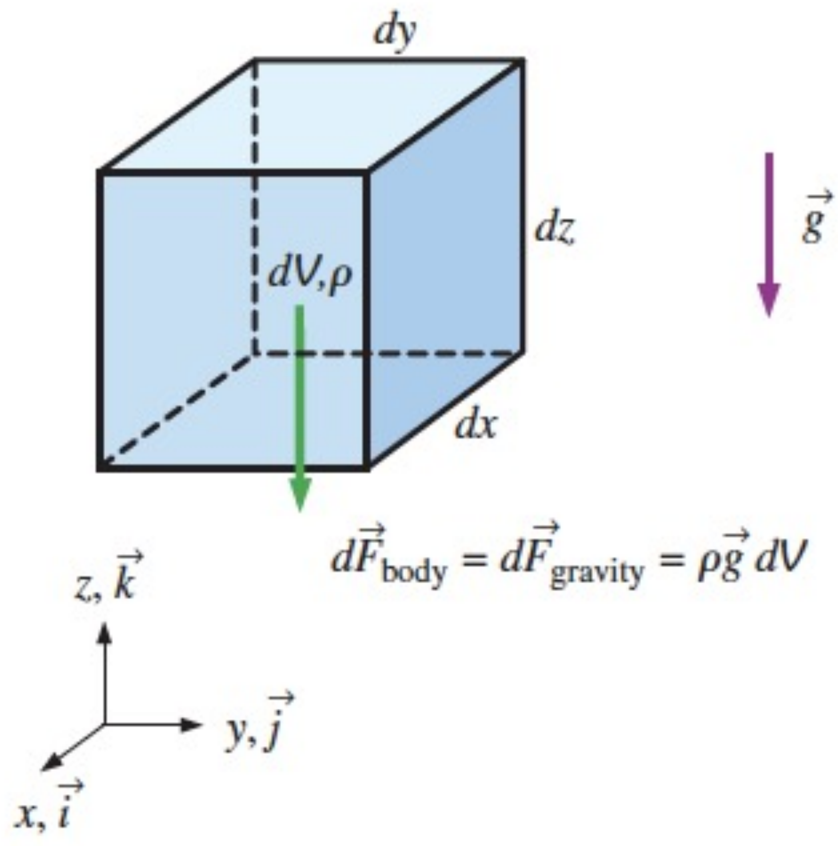
- References: chap. 5 and 6 Çengel, chap. 2 Faber and chap. 4 Acheson.
- Fluid dynamics deals with the relation between the motion of fluids considering the forces and moments which create the motion.
- Forces acting on fluid elements are of two types: body forces acting on the center of mass of the fluid element and surface forces acting through the surfaces.
- For ideal fluids (zero viscosity and compressibility) the surface forces reduce to an isotropic pressure (Pascal's Theorem) and the governing dynamical equation was derived by Euler.
- Euler's equation is a PDE that can be integrated along the streamlines and the integral is known as Bernoulli's equation (Bernoulli's first Theorem).

# Forces on a control volume

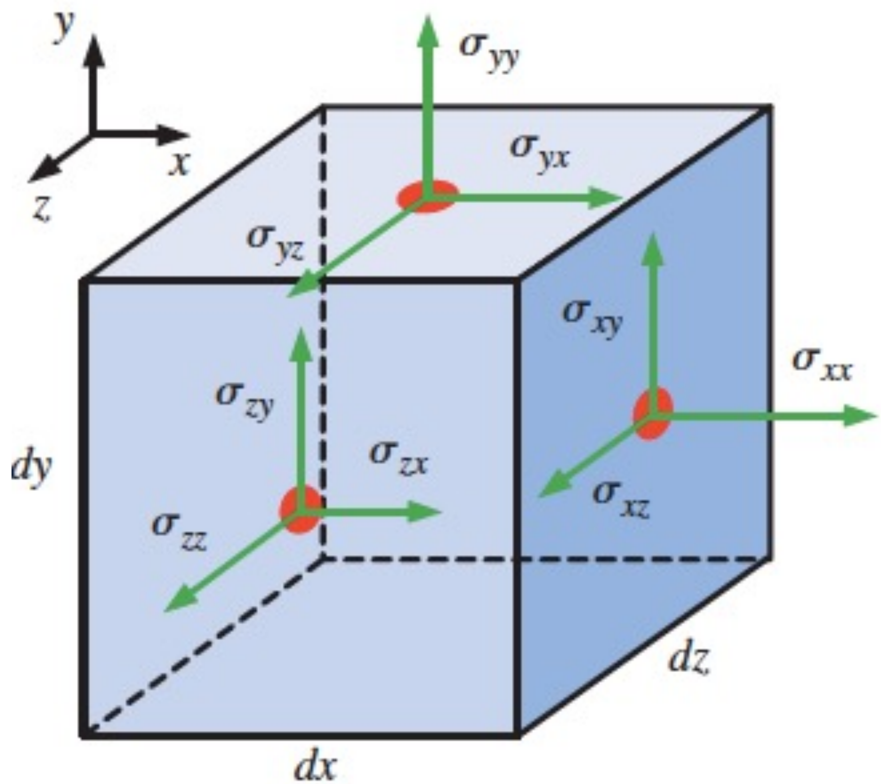



The forces acting on a control volume consist of body forces that act throughout the entire body of the control volume (such as gravity, electric, and magnetic forces) and surface forces that act on the control surface (such as pressure and viscous forces and reaction forces at points of contact).

# Body forces: 1 vector or rank 1 tensor



# Surface forces: 2 vectors or rank 2 tensor



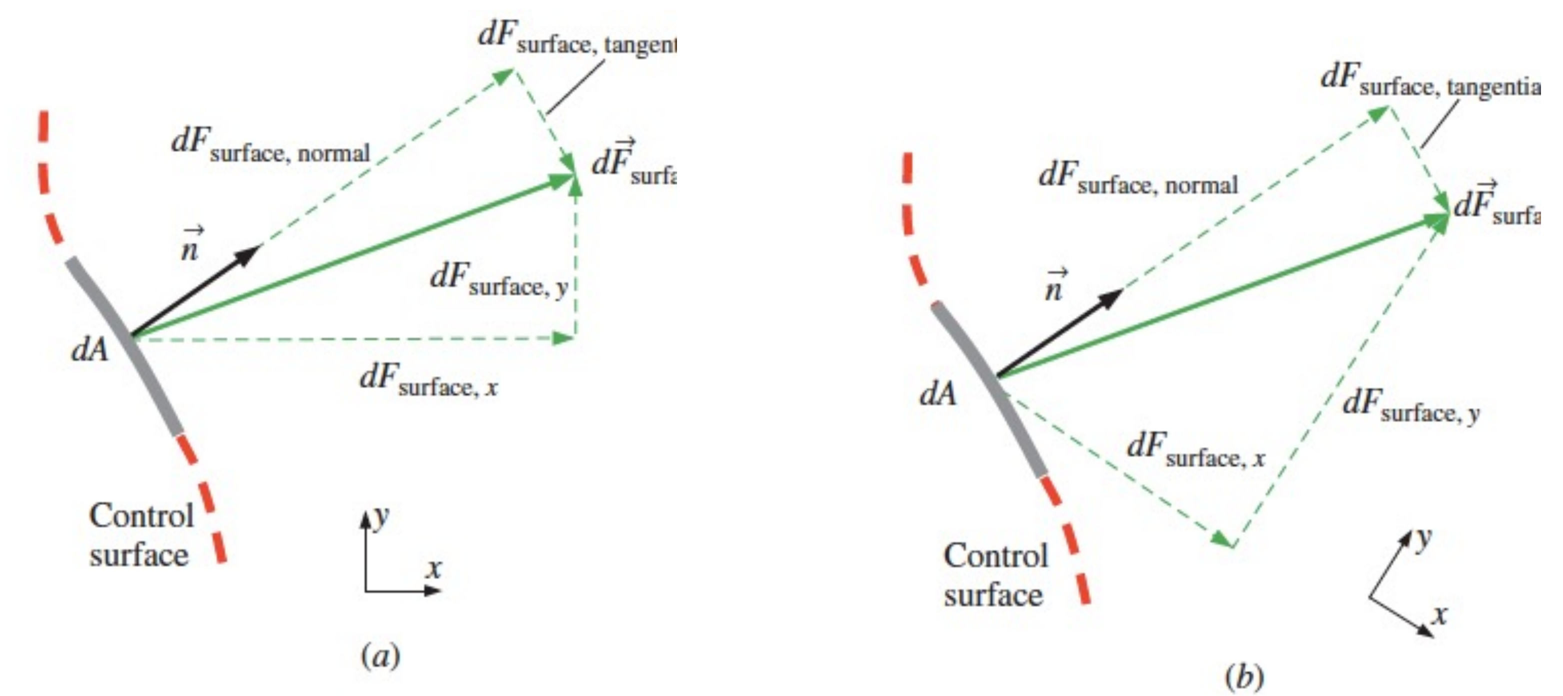

$\sigma_{ij}$ is the force per unit area in the direction of j through the plane (perpendicular to) i

# Stress tensor in cartesian coordinates

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

The diagonal components of the stress tensor are called normal stresses; they are composed of pressure (which always acts inwardly normal) and viscous stresses.
Viscous stresses are discussed in more detail later. The off-diagonal components, are called shear stresses; since pressure can act only normal to a surface, shear stresses are composed entirely of viscous stresses.

# Surface forces




*Surface force acting on a differential surface element:* $d\vec{F}_{\text{surface}} = \sigma_{ij} \cdot \vec{n}\, dA$

# Total force

Body forces act on each volumetric portion of the control volume. The body force acts on a differential element of fluid of volume dV within the control volume, and we must perform a volume integral to account for the net body force on the entire control volume.

$$\textit{Total body force acting on control volume:} \quad \sum \vec{F}_{\text{body}} = \int_{CV} \rho \vec{g}\, dV = m_{CV}\vec{g}$$

Surface forces act on each portion of the control surface. A differential surface element of area dA and unit outward normal $\vec{n}$ is shown, along with the surface force acting on it. We must perform an area integral to obtain the net surface force acting on the entire control surface.

$$\textit{Total surface force acting on control surface:} \quad \sum \vec{F}_{\text{surface}} = \int_{CS} \sigma_{ij} \cdot \vec{n}\, dA$$

# The linear momentum equation

$$\sum \vec{F} = \frac{d}{dt}\int_{\text{sys}} \rho \vec{V}\, d\forall$$

- Therefore, Newton's second law can be stated as the sum of all external forces acting on a system is equal to the time rate of change of linear momentum of the system.

- Applying the Reynolds transport theorem we find

$$\frac{d(m\vec{V})_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho \vec{V}\, d\forall + \int_{\text{CS}} \rho \vec{V}\,(\vec{V}_r \cdot \vec{n})\, dA$$

# RTT (recall)

For **moving** and/or **deforming** control volumes,

$$\frac{dB_{\text{sys}}}{dt} = \int_{\text{CV}} \frac{\partial}{\partial t}(\rho b)\, dV + \int_{\text{CS}} \rho b \vec{V} \cdot \vec{n}\, dA$$

$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho b\, dV + \int_{\text{CS}} \rho b \vec{V}_r \cdot \vec{n}\, dA$$

- Where the absolute velocity *V* in the second term is replaced by the **relative velocity** $V_r = V - V_{CS}$
- $V_r$ is the fluid velocity expressed relative to a coordinate system moving **with** the control volume.

# Newton's law for a control volume

*General*: $$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \int_{CS} \rho \vec{V}(\vec{V}_r \cdot \vec{n})\, dA$$

which is stated in words as

$$\begin{pmatrix}\text{The sum of all} \\ \text{external forces} \\ \text{acting on a CV}\end{pmatrix} = \begin{pmatrix}\text{The time rate of change} \\ \text{of the linear momentum} \\ \text{of the contents of the CV}\end{pmatrix} + \begin{pmatrix}\text{The net flow rate of} \\ \text{linear momentum out of the} \\ \text{control surface by mass flow}\end{pmatrix}$$

*Fixed CV*: $$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \int_{CS} \rho \vec{V}(\vec{V} \cdot \vec{n})\, dA$$

# Water jet on stationary plate

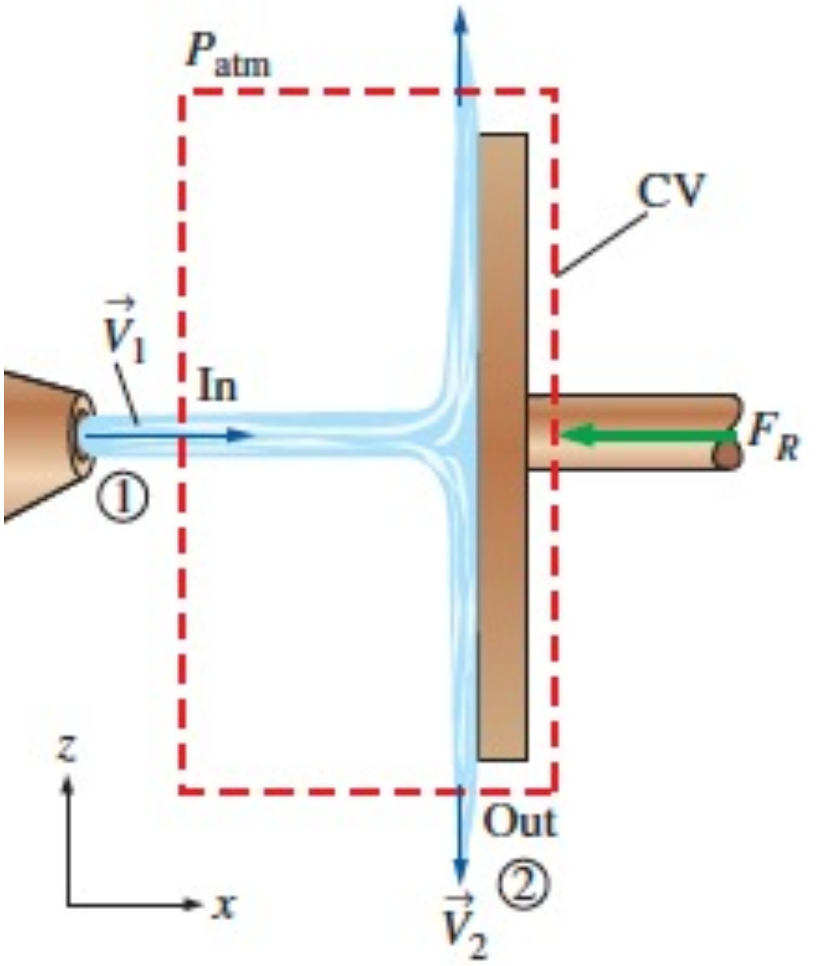
- The momentum equation for steady flow is given as

$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

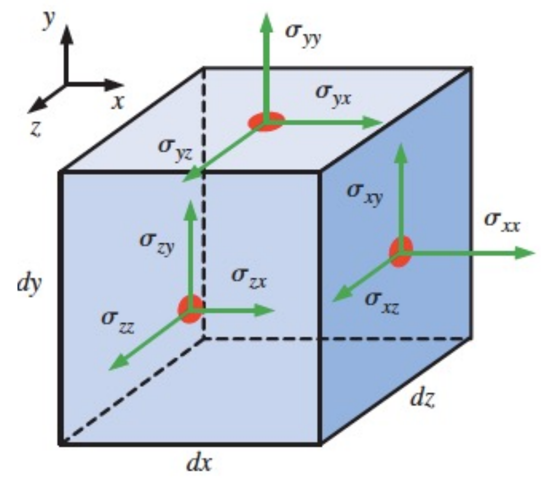
- The reaction force at the plate is (x-direction)

$$-F_R = 0 - \beta \dot{m} V_1$$

Note that β=1 in this course (incompressible fluid).

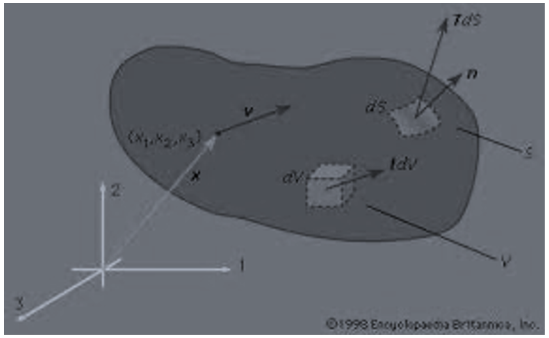


# Stress on surface element $dA$, with $n \equiv (n_x, n_y, n_z)$




From $t_i = \sigma_{ij} \cdot n_j$ we find the stress components:

- in the $x$ direction, $(\sigma_{xx}\, n_x + \sigma_{xy}\, n_y + \sigma_{xz}\, n_z)$
- in the $y$ direction, $(\sigma_{yx}\, n_x + \sigma_{yy}\, n_y + \sigma_{yz}\, n_z)$
- in the $z$ direction, $(\sigma_{zx}\, n_x + \sigma_{zy}\, n_y + \sigma_{zz}\, n_z)$

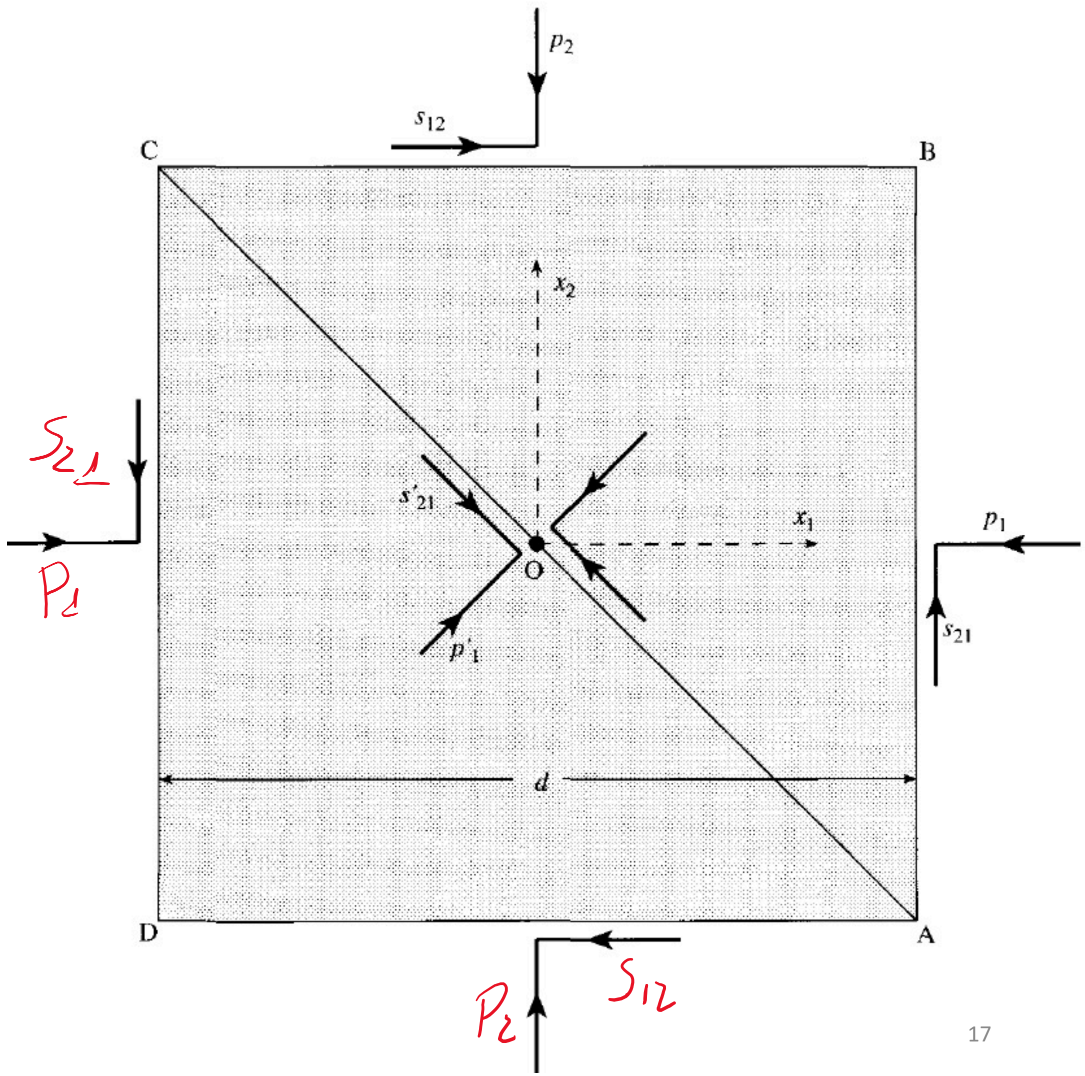



- The force is the product of the stress by the area.

# The stress tensor is symmetric

(Sec 1.3, Faber)

- Choose axes paralel to the directions of a cubic fluid element (dashed lines)
- Consider a cubic fluid element of side *d*
- Consider one face of the cube, *ABCD*. The plane that passes through *AB* has a normal in the *x (1)* direction.



# Balance of forces and torques

- Forces

Normal forces are: $d^2\, p_1$ , $d^2\, p_2$ and $d^2\, p_3$ with three other forces acting on the opposite faces. The forces act on an element of fluid of volume $d^3$. As $d \to 0$ the acceleration diverges as $\frac{1}{d}$ unless the forces on opposite faces of the cube balance.

- Torques

Tangencial forces: $d^2\, s_{21}$ and $d^2\, s_{12}$ produce a torque $d^3\,(s_{21} - s_{12})$ which produces an angular acceleration that diverges as $\frac{1}{d^2}$ (the moment of inertia of the element of fluid scales with the 5th power of *d* ) unless the torque vanishes. So $s_{21} = s_{12} = s_3$ (Faber's notation, 3 is the axis of rotation).

# Average pressure

- The average pressure is an invariant, i.e. it is the same for any rotation.

$$p = \frac{1}{3}(p_1 + p_2 + p_3)$$

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

# Fluids at mechanical equilibrium

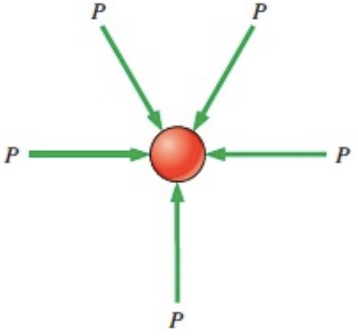
- At equilibrium the shear stresses vanish (otherwise there would be flow) which implies that,

$$p'_1 = p_1 = p_2,$$

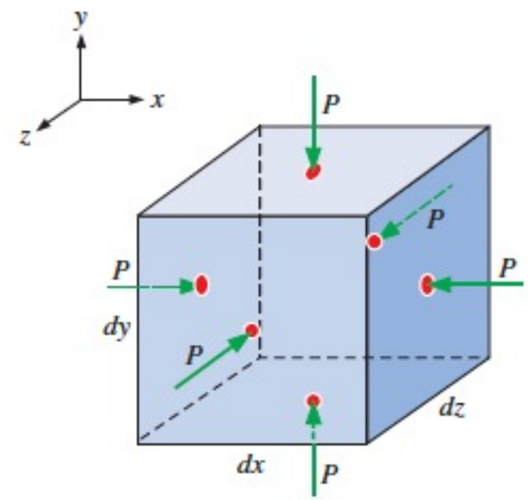
and similarly, for $p_3$. This means that $p_1 = p_2 = p_3 = p$ in any frame of reference, i.e. the pressure is a scalar field.

## Pascal's principle

The pressure is a scalar field, $p(\vec{r})$.

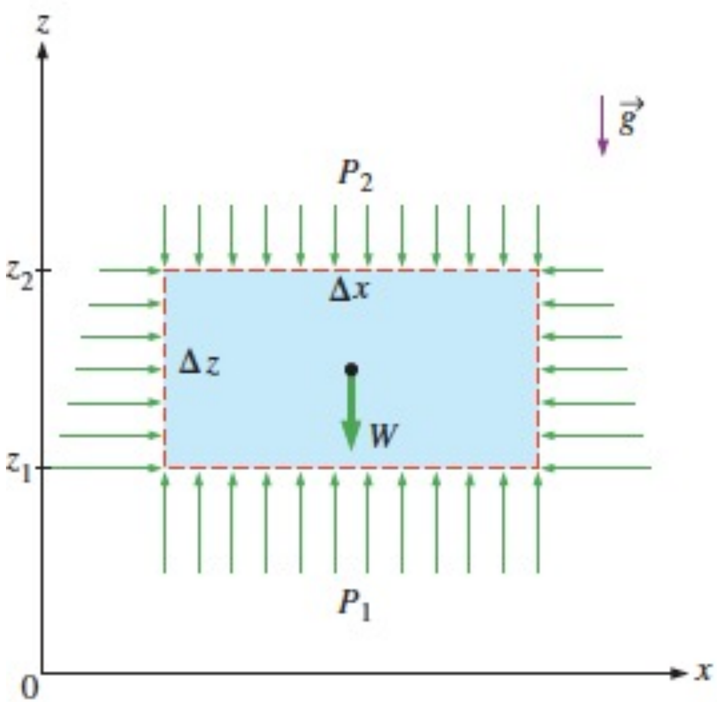


# Fluids at rest




$$\textit{Fluid at rest:} \qquad \sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}$$

# Variation of pressure with depth



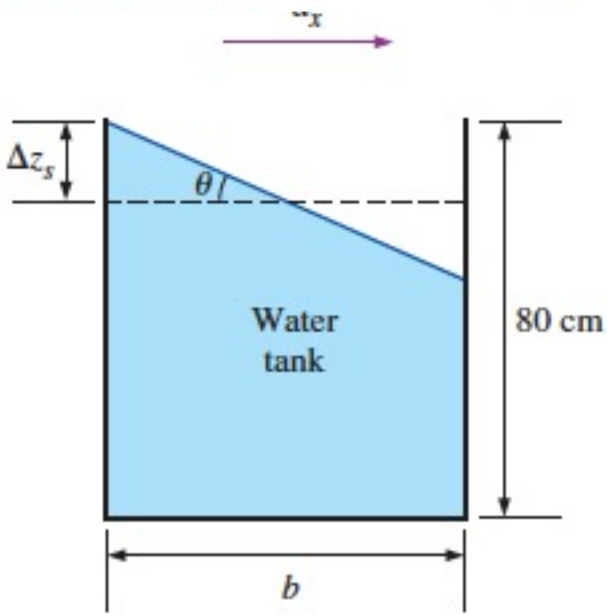

$$\Delta P = P_2 - P_1 = -\int_1^2 \rho g \, dz$$

$$P = P_{atm} + \rho g h \quad \text{or} \quad P_{gage} = \rho g h$$

# Rigid body motion I (overflow from a water tank)

From the previous example:

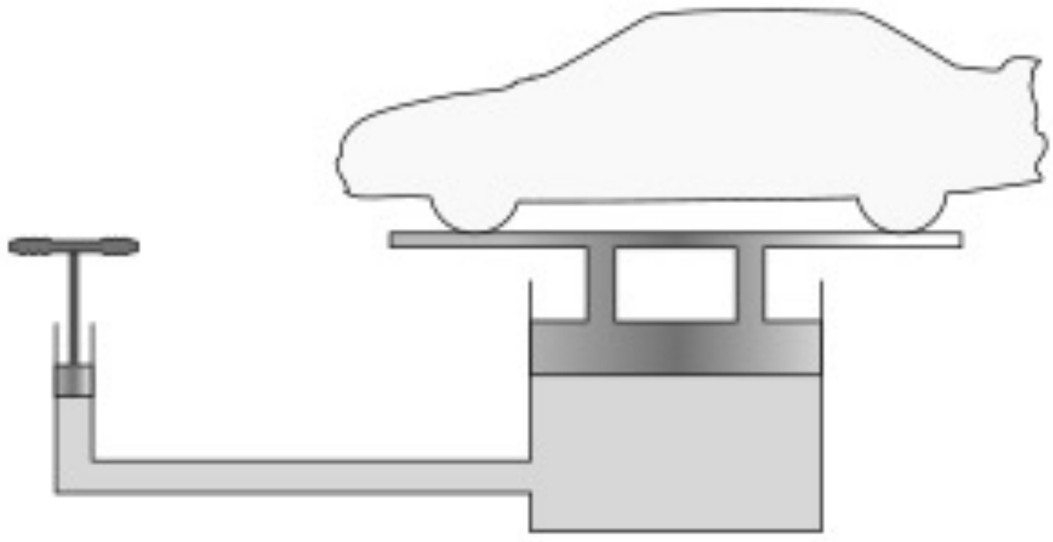
*Rigid-body motion of fluids:* $\vec{\nabla} P + \rho g \vec{k} = -\rho \vec{a}$




*Pressure variation:* $P = P_0 - \rho a_x x - \rho(g + a_z)z$

*Surfaces of constant pressure:* $\dfrac{dz_{\text{isobar}}}{dx} = -\dfrac{a_x}{g + a_z} = \text{constant}$

# Hydraulic press



A change in pressure at any point in an enclosed fluid **at rest** is transmitted undiminished to all points in the fluid.

# The Euler fluid: zero viscosity and zero compressibility

As a result the shear stresses are zero and the density is constant

# Euler fluid

(Acheson, chap. 1)

- For an Euler fluid the continuity equation implies that $\nabla \cdot \boldsymbol{u} = 0$

and the inviscid (zero viscosity) condition implies that the stress tensor reduces to a scalar isotropic pressure, *p,* which may vary in space (pressure field).

- The surface forces acting on an element of fluid, per unit volume, are given

by $-\nabla p$ (recall that the force in the x direction is $-\frac{\partial p}{\partial x}$).

- The forces per unit mass are then $-\frac{\nabla p}{\rho}$.

- The total force may include body terms, such as gravity, $-\nabla gz$.

- The Euler equation is

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \boldsymbol{g},$$

# Euler fluid

- 4 equations (continuity + Euler) and 4 unknowns (u, v, w, p);

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,$$

$$\nabla \cdot \vec{u} = 0 \Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

- The gravitational force, being conservative, can be written as the gradient of a potential (=gz):

$$\boldsymbol{g} = -\nabla\chi. \quad = gz$$

# Euler fluid

- Euler's equation becomes:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \chi\right)$$

Identity

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = (\nabla \wedge \boldsymbol{u}) \wedge \boldsymbol{u} + \nabla(\tfrac{1}{2}\boldsymbol{u}^2)$$

- We can write:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\nabla \wedge \boldsymbol{u}) \wedge \boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \tfrac{1}{2}\boldsymbol{u}^2 + \chi\right)$$

# Bernoulli streamline theorem

- If the fluid is steady,

$$(\nabla \wedge \boldsymbol{u}) \wedge \boldsymbol{u} = -\nabla H$$

where: $$H = \frac{p}{\rho} + \tfrac{1}{2}\boldsymbol{u}^2 + \chi.$$

- On taking the dot product with u, we obtain

$$(\boldsymbol{u} \cdot \nabla) H = 0,$$

- If an ideal fluid is in steady flow, then H is constant along a streamline.
- The above theorem says nothing about H being the same constant on different streamlines.

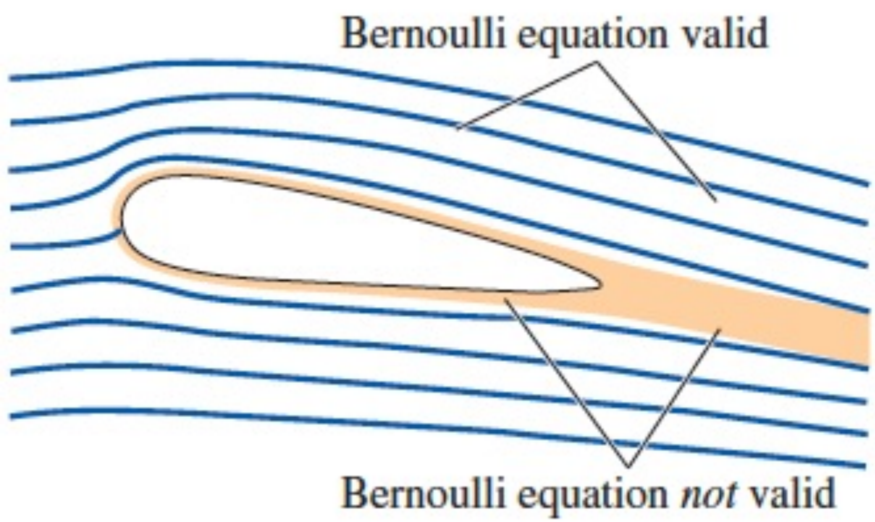
# Bernoulli theorem for irrotational flow

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\nabla \wedge \boldsymbol{u}) \wedge \boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \tfrac{1}{2}\boldsymbol{u}^2 + \chi\right)$$

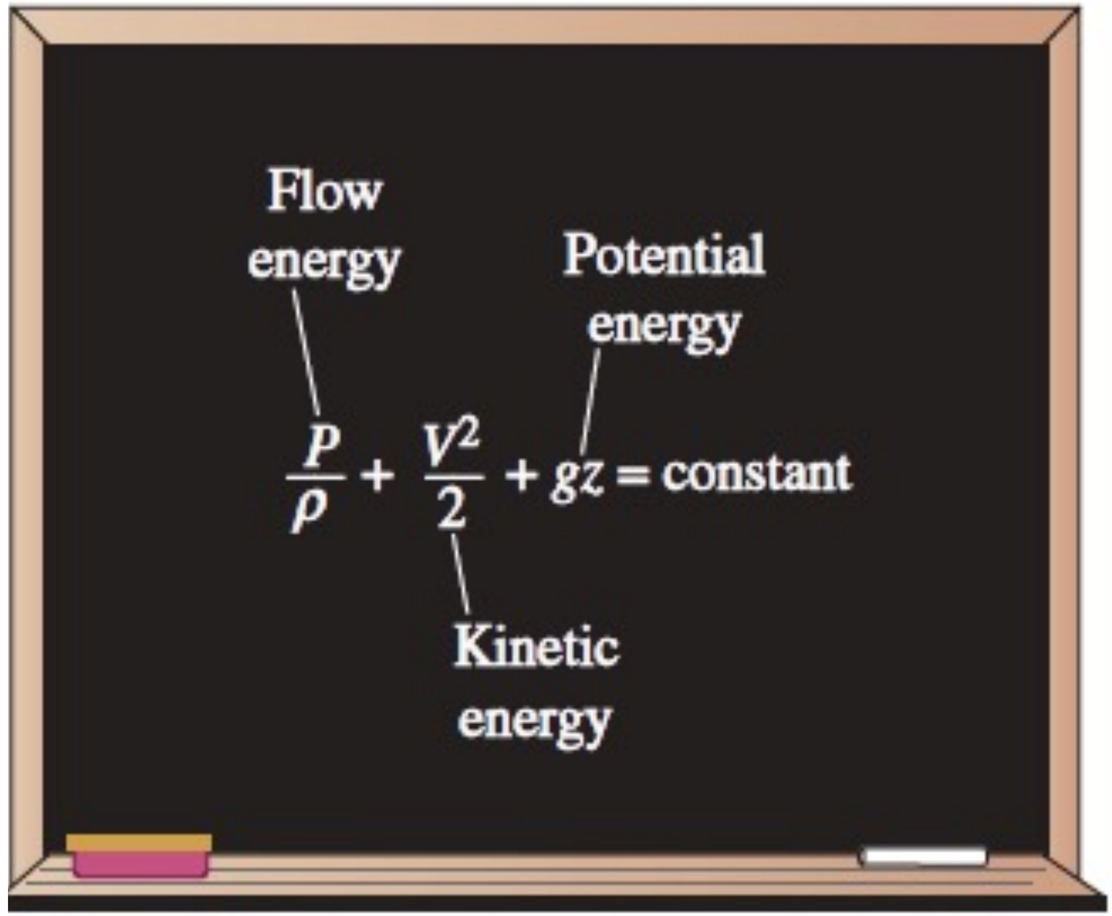
If an ideal fluid is in steady irrotational flow, then H is constant throughout the whole flow field.

The Bernoulli equation is an approximate equation that is valid only in inviscid regions of flow where net viscous forces are negligibly small compared to inertial, gravitational, or pressure forces.

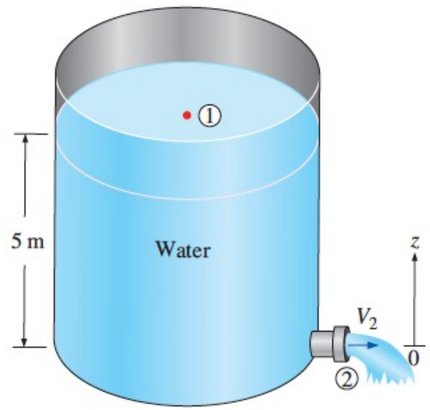
Such regions occur outside of boundary layers and wakes.



The sum of the kinetic, potential, and flow energies of a fluid particle is constant along a streamline, during steady flow, when compressibility and frictional effects are negligible.



# Velocity of discharge from a large tank




Direct application of Bernoulli's equation,

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g}^{\approx 0} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2^{\,0} \quad \rightarrow \quad z_1 = \frac{V_2^2}{2g}$$

The discharge velocity is,

$$V_2 = \sqrt{2gz_1}$$

# Table top experiment

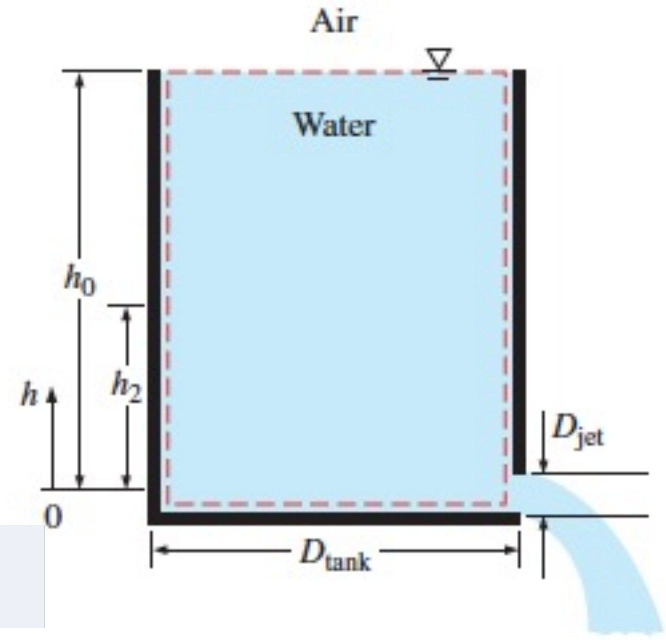
Try to check this for water and oil.

# Time of discharge of a tank

$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm_{CV}}{dt}$$

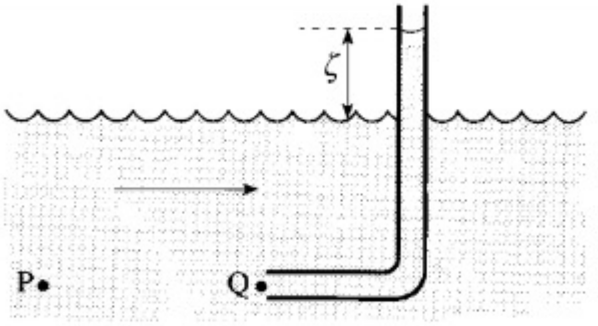
$$\dot{m}_{out} = (\rho VA)_{out} = \rho\sqrt{2gh}A_{jet}$$

$$-\rho\sqrt{2gh}A_{jet} = \frac{d(\rho A_{tank}h)}{dt} \rightarrow -\rho\sqrt{2gh}(\pi D_{jet}^2/4) = \frac{\rho(\pi D_{tank}^2/4)dh}{dt}$$



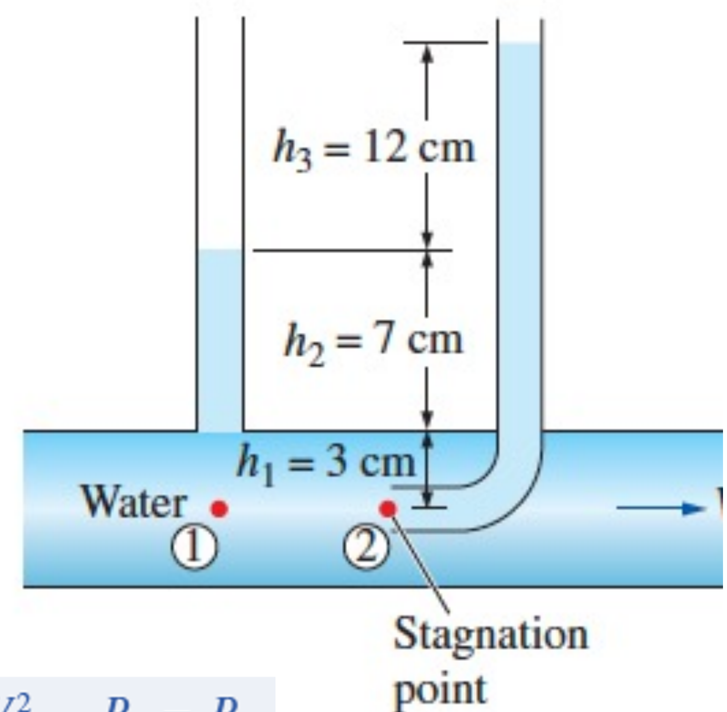

$$\int_0^t dt = -\frac{D_{tank}^2}{D_{jet}^2\sqrt{2g}}\int_{h_0}^{h_2}\frac{dh}{\sqrt{h}} \rightarrow t = \frac{\sqrt{h_0} - \sqrt{h_2}}{\sqrt{g/2}}\left(\frac{D_{tank}}{D_{jet}}\right)^2$$

# Pitot tube




Location of pitot tubes on a Boeing 777

# Pitot tube




Hydrostatic pressure at points 1 and 2,

$$P_1 = \rho g(h_1 + h_2)$$

$$P_2 = \rho g(h_1 + h_2 + h_3)$$

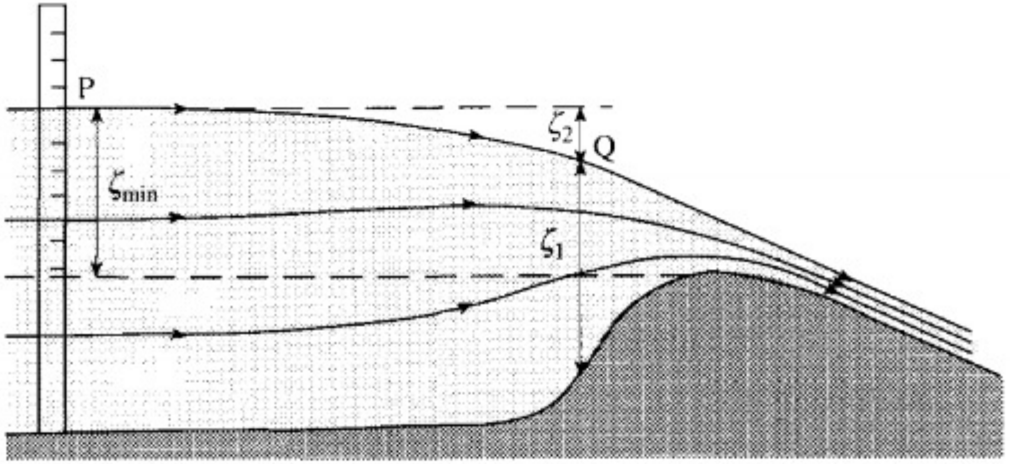
Application of Bernoulli's equation,

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + \cancel{z_1} = \frac{P_2}{\rho g} + \cancelto{0}{\frac{V_2^2}{2g}} + \cancel{z_2} \rightarrow \frac{V_1^2}{2g} = \frac{P_2 - P_1}{\rho g}$$

Gives for the for the velocity at 1,

$$V_1 = \sqrt{2gh_3}$$

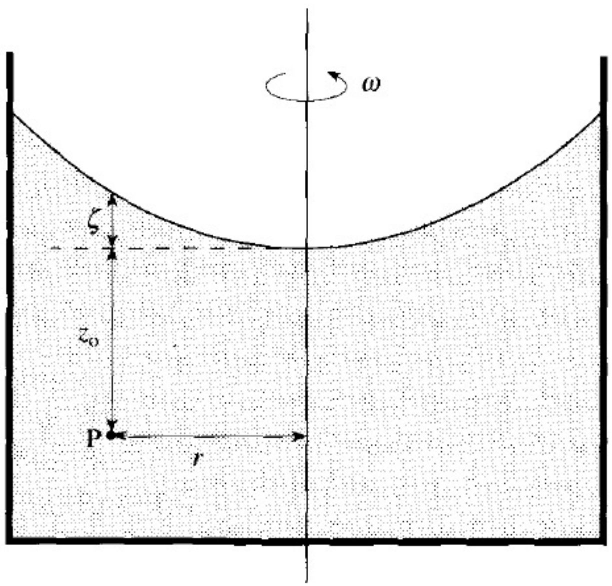
# Faber 2.10



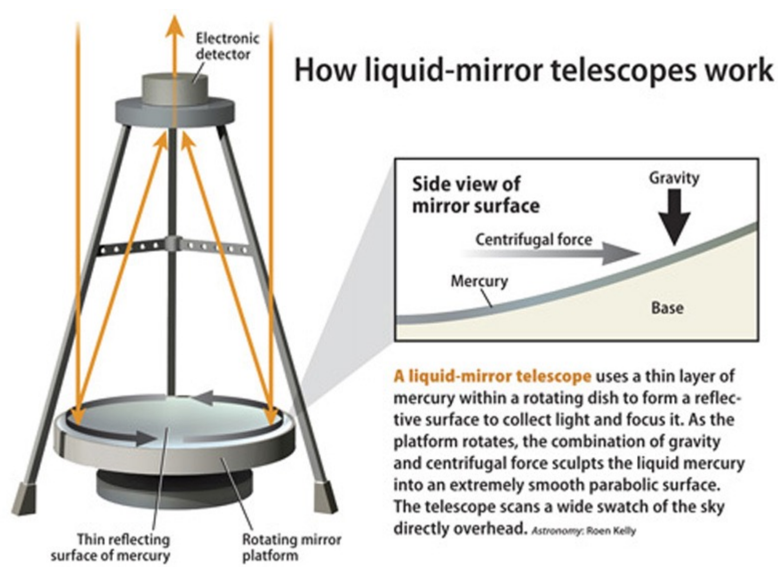

Calculate the flow rate as a function of $\zeta_{min}$

# The bucket of liquid

Faber, 2.5



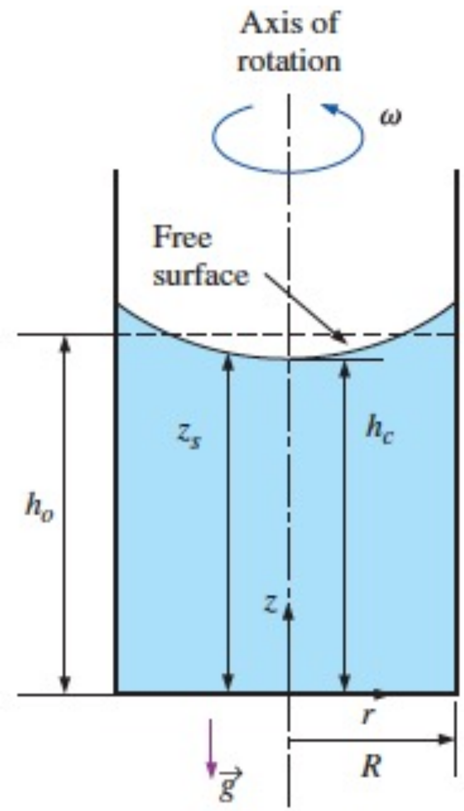

Liquid mirror telescope:
https://www.youtube.com/watch?v=Q5Cr9P-Q88Y




1) Find the equation for the free surface.

2) Find the pressure for a constant height.

# Rigid body motion II (free surface of a rotating vertical cylinder)




In cylindrical coordinates,

$$\frac{\partial P}{\partial r} = \rho r \omega^2, \quad \frac{\partial P}{\partial \theta} = 0, \quad \text{and} \quad \frac{\partial P}{\partial z} = -\rho g$$

and

$$dP = \rho r \omega^2 \, dr - \rho g \, dz$$

From *dP=0*, we find the isobars

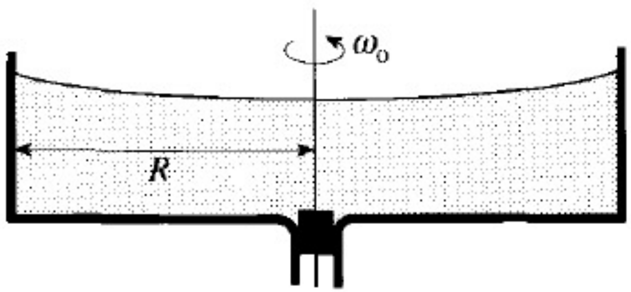
*Surfaces of constant pressure*: $$z_{\text{isobar}} = \frac{\omega^2}{2g} r^2 + C_1$$

Mass conservation yields, $h_c = h_0 - \dfrac{\omega^2 R^2}{4g}$

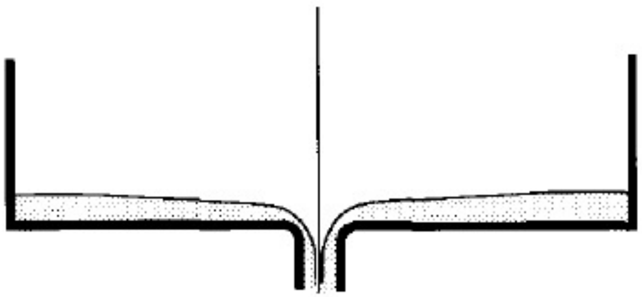
*Free surface*: $$z_s = h_0 - \frac{\omega^2}{4g}(R^2 - 2r^2)$$

# The plug-hole vortex (Faber)

Conservation of angular momentum:
particles initially at *R* move to *r,* with angular velocity $\omega \approx \omega_0 \dfrac{R^2}{r^2}.$



(a)


(b)

Using the expression for the transverse pressure gradient,

$$\frac{\partial p}{\partial r} \approx \rho\omega^2 r \approx \frac{\rho\omega_0^2 R^4}{r^3},$$

Integration, yields for the depth at *r* (measured from the height at R):

$$\zeta \approx \frac{\omega_0^2 R^4}{2r^2 g}.$$

Both the efflux of the water and the trajectory of the resulting jet are well described by ideal fluid theory

Another spectacular success is the theory of flight. The ideal flow of air around a wing is able to describe the lift necessary for flight, and much more.